Effective permittivity and permeability of coated metal powders at microwave frequency

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Abstract

Recent developments of the microwave heating of compacted metal powders bring a demand on calculation of corresponding effective optical parameters. Since the existing models cannot be applied to such powders, in this paper new effective dielectric permittivity and magnetic permeability are introduced by means of the combination of Mie theory and Bruggeman’s effective medium. It is demonstrated that both an insulating shell and the skin effect should be taken into account. The shell drastically modifies the effective permittivity, while ignoring the skin effect brings to the permittivity an error as large as 30% at 30 GHz for copper powder with grains of 50 μm radius. The induced currents are shown to affect the magnetic properties of the powders. The strong skin effect substantially reduces the permeability of magnetic grains, and thus the intrinsic permeability measured in experiments can be much less than the real one. The strong dependence of the effective parameters on the microstructure of powders suggests a variety of possible heating scenarios in experiments.

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Recently, the utilization of microwaves for heating was extended to compacted metal powders [1]. Bulk metal is known to mainly reflect microwaves since an alternating incident electromagnetic field induces macroscopic current, which in turn generates a strong secondary field oppositely directed to the incident one. In many metal powders irradiated by microwaves, macroscopic currents are not induced since the conductive grains are coated by an insulating oxide layer. As a result, microwaves can penetrate deeply inside the sample and induce microscopic currents strong enough to heat the cores by means of the Joule heating.

Though the microstructure of a composite determines its macroscopic properties, sometimes this information is complicated or may not be available. However, composites frequently behave like a homogeneous material on the scales relevant to an experiment. In this case, approximate homogeneous effective parameters are introduced.

There are different ways to introduce the effective parameters [2–6]. Among them the effective-medium approximation (EMA) [7] is frequently used. A number of numerical and experimental works prove that EMA is a good approximation for the estimations of optical functions of aggregate composites [8–10]. In this approach, the interactions between grains are taken into account by self-consistent manner and thus a multi-body problem is reduced to a single body one. In particular, each constituent of the powder along with pores is considered as equivalent inclusions embedded in effective Bruggeman’s medium. Shape of the inclusions is usually chosen to be spherical, but it also can take other convenient forms. Parameters of the medium are determined self-consistently. For example, the space averaged far electromagnetic field due to these inclusions is required to vanish.

In this study expression for the far field is obtained by means of Mie theory [11]. The Mie theory (or sometimes Lorenz–Mie–Debye theory) is a general analytical solution for the electromagnetic field inside and outside of a homogeneous and isotropic spherical particle irradiated by a plane wave. The solution is given in terms of infinite series. Further improvements of the theory extend it to the case of coated particles [12]. Due to the use of Mie theory, the magnetic response is assumed to be isotropic and linear. Questions related the microwave heating of materials with spontaneous magnetization are answered by Tanaka et al. [13].

The obtained expressions for the effective parameters take into account the eddy current and an insulating layer simultaneously, and this differs from the expressions in an existing literature.

This paper is mainly focused on the case of a thin shell compared with the radius of the core, which frequently appears in experiments. In this case, a strong effect of the shell on the
The effective permittivity, and thus on the heating rate, is not a priori apparent. Also it is not clear how the finite wavelength of microwaves affects the effective dielectric permittivity. Indeed, an incident electromagnetic wave induces microscopic currents in grains which are known to cause magnetization and results in the magnetic permeability of the powder. At the same time, the relationship between the effective permittivity and the induced currents is not evident.

Another question to be considered is the influence of the skin effect on the effective permittivity of nonmagnetic or magnetic coated powders. Specifically, does the shell affect the magnetization of a nonmagnetic powder? What is the overall effect of the induced current and the intrinsic permeability on the effective permeability? These questions are to be answered in the following sections.

1. Theoretical background

To determine the effective optical parameters of metal powders, let us consider a spherical inclusion embedded into homogeneous medium with yet unknown dielectric permittivity \( e_{\text{eff}} \) and magnetic permeability \( \mu_{\text{eff}} \). The inclusion is subjected to a plane time-harmonic electromagnetic wave (Fig. 1). Bruggeman’s approximation states that in effective medium the average scattered field due to inclusions must vanish.

Components of the scattered field can be found by means of Mie theory [11]. Since in microwave heating experiments the grain size in powders is usually much smaller than the wavelength of microwaves in background medium, only the first term of the Mie series contributes to the solution. In this case, the components of the scattered far field \( E_{s} \) at distance \( r \) from the particle (\( k_{\text{eff}}r \gg 1 \)) are given in spherical coordinates \((r, \theta, \phi)\) by

\[
E_{s} = E_{0} e^{-i\omega t} \frac{3i e^{ik_{\text{eff}}r}}{2 k_{\text{eff}}r}(a_{1} + b_{1} \cos \theta) \cos \phi ,
\]

where \( E_{0} \) is the amplitude of the incident electric field at the inclusion position; \( i = \sqrt{-1} \); \( \omega = 2\pi f \) is the frequency of the electromagnetic wave; \( k_{\text{eff}} = \omega / \sqrt{\varepsilon_{\text{eff}} \mu_{\text{eff}}} \); \( a_{1} \) and \( b_{1} \) are the Mie coefficients to be determined below. The self-consistent condition yields

\[
\langle a_{1} \rangle = 0, \quad \langle b_{1} \rangle = 0 ,
\]

where \( \langle \cdot \rangle \) denotes averaging over various inclusions.

When the inclusion is a coated particle, the Mie theory gives the following coefficients \( a_{1} \) and \( b_{1} \) (see Appendix):

\[
a_{1} \approx \frac{2\pi \varepsilon_{p} - \varepsilon_{\text{eff}}}{3 \left( \mu_{p} + 2\mu_{\text{eff}} \right)},
\]

\[
b_{1} \approx \frac{2\pi \varepsilon_{p} - \varepsilon_{\text{eff}}}{3 \left( \mu_{p} + 2\mu_{\text{eff}} \right)},
\]

with

\[
a_{1} \approx \frac{2\pi \varepsilon_{p} - \varepsilon_{\text{eff}}}{3 \left( \mu_{p} + 2\mu_{\text{eff}} \right)},
\]

for highly conducting nonmagnetic materials \( k_{1} r_{1} = (1+i) r_{1} / \delta \), where

\[
\delta = \sqrt{\frac{2}{\omega \varepsilon_{\text{eff}} \mu_{0}}}
\]

is the skin depth (\( \mu_{0} = 1.2566 \times 10^{6} \) H/m is the permeability of free space, \( \sigma \) is the static conductivity), and thus the factor \( F_{0} \) takes into account the skin effect. When the skin effect is negligible (\( r_{1} / \delta \ll 1 \)), the factor becomes \( F_{0} = 1 \).

The assumptions used during derivation of Eqs. (4) and (5) are \( |k_{\text{eff}}r_{2}| \ll 1 \) and \( |k_{2}r_{2}| \ll 1 \). Note that the magnitude of \( |k_{1}r_{1}| \) is still unconstrained, and thus these expressions can be used for a dielectric core as well as for a highly conductive one.

When the inclusion is a pore filled by a gas with the permittivity \( \varepsilon_{g} \) and permeability \( \mu_{g} \), the coefficients \( a_{1} \) and \( b_{1} \)
are readily given by Eqs. (4) and (5) with \( \varepsilon_p \) and \( \mu_p \) replaced by \( \varepsilon_g \) and \( \mu_g \), respectively.

Finally, from Eq. (3), the effective permeability \( \mu_{\text{eff}} \) and permittivity \( \varepsilon_{\text{eff}} \) of Bruggeman’s medium are determined by solving the equations

\[
p\frac{\mu_p - \mu_{\text{eff}}}{\mu_p + 2\mu_{\text{eff}}} + (1 - p)\frac{\mu_g - \mu_{\text{eff}}}{\mu_g + 2\mu_{\text{eff}}} = 0, \tag{11}
\]

\[
p\frac{\varepsilon_p - \varepsilon_{\text{eff}}}{\varepsilon_p + 2\varepsilon_{\text{eff}}} + (1 - p)\frac{\varepsilon_g - \varepsilon_{\text{eff}}}{\varepsilon_g + 2\varepsilon_{\text{eff}}} = 0. \tag{12}
\]

Here \( p \) is the volume fraction of particles.

The validity region of the obtained expressions is outlined by the more stringent assumption, namely \( |\varepsilon_{\text{eff}}| < 1 \) and \( |\mu_{\text{eff}}| < 1 \). The lower limit comes from physical constraints. The grains should be large enough to use the macroscopic conductivity, permittivity and permeability. At microwave frequencies, metal powders with grains of a few tens of microns or less usually fall between these limits.

Eqs. (11) and (12) can be reduced to those derived in other works. First, when \( r_2 = r_1 \) and \( \mu_2 = \mu_1 \), then one obtains \( \mu_{\text{eff}} = F_0 \mu_1 \), which is the known result for the magnetic permeability of a homogeneous metal particle [14–16,8]. In this case, the induced eddy current causes magnetization of the particle, and therefore the effective relative permeability is not unity in spite of the fact that the particle itself may be nonmagnetic. Secondly, the obtained expressions correctly describe the optical behavior of dielectric coated particles (\( F_0 = 1 \)) considered by Bohren et al. [17] and Buchelnikov et al. [18]. In the latter work, some comparison of the obtained effective permittivity with the existing expressions is made. In contrast to the works noted above, expressions (11) and (12) are more general since they take into account the skin effect and the insulating shell simultaneously.

It is interesting to note that, from Eq. (6) in the case of a homogeneous grain when \( r_2 = r_1 \) and \( \varepsilon_2 = \varepsilon_1 \), the “equivalent permittivity” of the grain becomes \( \varepsilon_{\text{eff}} = \varepsilon_1 F_0 \) [19], though it is expected to be \( \varepsilon_{\text{eff}} = \varepsilon_1 \). At microwave frequencies the imaginary part of the dielectric permittivity of a highly conductive core is much larger than the real one, and in this study the real part is neglected, i.e. \( \varepsilon_1 \approx \iota \delta /\varepsilon_0 \). The factor \( F_0 \) introduces a large and negative part to the permittivity \( \varepsilon_{\text{eff}} \) which corresponds to free electron dielectric response at microwave frequencies in good conductors according to Drude and Sommerfeld models [20]. For example, for a copper grain of \( r_1 = 5 \mu \text{m} \) irradiated at 2.45 GHz the intrinsic permittivity is \( \varepsilon_1 /\varepsilon_0 = 1.43 \times 10^8 \), while one has \( F_0 \varepsilon_1 /\varepsilon_0 = (-1.1 + 1i 2.1) \times 10^8 \), where \( \varepsilon_0 = 8.85 \times 10^{-12} \text{F/m} \) is the permittivity of free space.

In this study, the optical parameters of gas filling pores are assumed to be \( \varepsilon_g = \varepsilon_0 \) and \( \mu_g = \mu_0 \).

2. Effective permittivity

2.1. Effect of skin depth

With the Mie expressions the effect of skin depth on the effective permittivity \( \varepsilon_{\text{eff}} \) can be naturally taken into account. This effect was usually ignored in previous works. Numerical analysis shows that the estimation error caused by exclusion of the skin effect, defined by \( |\varepsilon_{\text{eff}} - \varepsilon_{\text{eff}}(F_0 = 1)|/\varepsilon_{\text{eff}} \), increases with increasing frequency and the particle core radius, and decreases as the core conductivity increases (Fig. 3). Ignoring the skin effect introduces

**Fig. 3.** Estimation error in \( \varepsilon_{\text{eff}} = \varepsilon_1 + \iota \delta \) caused by ignoring the skin effect: (a) dependence on frequency and (b) dependence on the core radius. Here, \( \Delta r = r_2 - r_1 \), \( \sigma_{\text{Cu}} = 5.8 \times 10^7 \text{S/m} \), \( \varepsilon_1 /\varepsilon_0 = 3 + \iota 0.3 \), \( p = 0.9 \). The relative error is defined by \( |\varepsilon_{\text{eff}} - \varepsilon_{\text{eff}}(F_0 = 1)|/\varepsilon_{\text{eff}} \). Largest of errors of \( \varepsilon_{\text{eff}} \) and \( \delta_{\text{eff}} \) is shown.

**Fig. 4.** Effect of the shell thickness on effective permittivity \( \varepsilon_{\text{eff}} \) for \( \nu_2 /\nu_1 = 3 + \iota 0.3 \), \( r_1 = 5 \mu \text{m} \), \( p = 0.9 \), \( \sigma = 5.8 \times 10^7 \text{S/m} \) and \( f = 2.45 \text{GHz} \). The shell thickness \( \Delta r \) changes from 0.1 to 100 nm. Maximal values of the real and imaginary parts of the effective permittivity \( \varepsilon_{\text{eff}} /\varepsilon_0 \) are \( 7.7 \times 10^7 \) and \( 1.7 \times 10^8 \), respectively. The results demonstrate that even a thin shell drastically reduces \( \varepsilon_{\text{eff}} \).
an error as large as tens of percents for compacted powders with
grains of a few tens of microns or larger at \( f = 30 \text{ GHz} \) [Fig. 3(a)]. In
the case of finer powders with a few microns radii, the influence of the skin effect on \( e_{\text{eff}} \) is small [Fig. 3(b)]. Smaller thickness of the shell also magnifies the error.

2.2. Effect of shell thickness

In practice the radius of metal core \( r_1 \) is frequently much larger than the thickness of shells, i.e. \( \Delta r = r_2 - r_1 \ll r_1 \), which raises a question of the influence of the shell on the effective parameters. Nevertheless, the calculations show that even a very thin shell \( \Delta r/r_1 < 1 \times 10^{-2} \) significantly reduces \( e_{\text{eff}} \) in comparison with the permittivity of the core (Fig. 4).

When the shell is thick, the effective permittivity tends to the permittivity of the shell \( e_2 \). In the opposite case when the grains in a compacted powder are almost homogeneous, \( e_{\text{eff}} \) is determined by the permittivity of the core \( e_1 \). Since the difference between \( r_1 \) and \( r_2 \) can be many orders of magnitude, the effective permittivity changes drastically with the thickness of the shell.

3. Effective permeability of nonmagnetic powders

3.1. Effect of skin depth

When an uniform nonmagnetic metal particle \((\mu_1 = \mu_2 = \mu_0)\) is subjected to microwave irradiation, an induced current causes magnetic moment and the particle effectively behaves like a magnetic one [14]. For particles coated by an insulating shell, electric current flows only in the core and thus “permeability of grains” \( \mu_p = \mu_0 \mu_0 \) is determined merely by the core parameter \( r_1/\delta \) (Fig. 5).

The result shows that the particles do not couple with the magnetic component when the grain cores are very small \((\mu_p \to \mu_0)\) when \( r_1/\delta \ll 1 \) or very large \((\mu_p \to 0\) when \( r_1/\delta \gg 1 \). Between these cases, there is a region where the imaginary part of the “permeability of grain” \( \mu_p \) has a maximum. The flow of the induced current changes here from volumetric to surface one [14].

Clearly, the powder composed of such particles also exhibits magnetic behavior. Position of \( \mu_{\text{eff}} \) maximum directly corresponds to that of \( \mu_p \) and it is almost independent of the particles volume fraction. Of course, the volume fraction should be high enough for the pronounced induced effective permeability.

3.2. Effect of shell thickness

In contrast to the effective permittivity, the effective permeability \( \mu_{\text{eff}} \) is almost insensitive to the shell thickness when \( \Delta r \ll r_1 \) (Fig. 6). The magnetic permeability is caused by induced currents in a core, and therefore parameters of the core mainly determine it. Nevertheless, in the limiting case when the thickness of the shell is comparable or larger than the radius of the core, the effective permeability tends to the permeability of the shell (not shown in the figure).

4. Effective permeability of magnetic powders

The skin effect is known to modify the effective permeability of powders made of homogeneous grains [15,16,8]. In the case of coated grains, a shell contributes to the effective magnetic permeability along with a core, and a question arises about the significance of this contribution. Another question concerns with interaction of the induced permeability with the intrinsic one. To answer these questions, two cases are investigated: (i) grains consist of nonmagnetic cores covered by magnetic shells and (ii) grain cores are magnetic but coated by nonmagnetic shells.

In order to reduce the number of independent parameters, the “permeability of grain” \( \mu_p \) is considered instead of the effective permeability \( \mu_{\text{eff}} \). To distinguish the induced permeability determined by Eq. (6) with \( \mu_1 = \mu_2 = \mu_0 \) from the intrinsic one, the former is denoted as \( \mu_{\text{ind}} \) in this section.

In following calculations, the ratio \((r_1/r_2)^3\) is fixed at 0.9. The ratio \( r_1/\delta \) is set to 0.1, 2 and 5, which corresponds to nonmagnetic, highly lossy and highly diamagnetic behavior of \( \mu_{\text{ind}} \) (see Fig. 5, where \( \mu_p \equiv \mu_{\text{ind}} \)). The range of the intrinsic permeability is chosen to demonstrate the behavior of \( \mu_p \).

Dependence of the magnetic “permeability of grain” \( \mu_p \) on that of the shell \( \mu_2 \) in the case of nonmagnetic core is presented in Fig. 7. The results show that the shell weakly affects the \( \mu_p \) for the presented conditions, which is contrasted to the case of dielectric.

![Fig. 6. Effect of the shell thickness on effective permeability \( \mu_{\text{eff}} \) for \( r_1 = 5 \mu m, p = 0.9, \sigma = 5.8 \times 10^7 \text{ S/m} \) and \( f = 2.45 \text{ GHz} \). The shell thickness \( \Delta r \) changes from 1 to 100 nm. For the presented parameters, the effective \( \mu_{\text{eff}} \), on the contrary to \( e_{\text{eff}} \), is almost insensitive to the thickness of shell.](image-url)
permittivity. Note that the permeability $\mu_p$ can be roughly decomposed into the $r_1/\delta$ depending part and the $\mu_2$ depending part. It means that the shell and the core interact with the magnetic field almost independently. This conclusion is supported by the empirical expressions for $\mu_p$ obtained based on Fig. 7:

$$
\mu_p \approx \mu_{ind} \frac{r_1^3}{1 + 2 \mu_2} \frac{1 - (r_1/r_2)^3}{2 + (r_1/r_2)^3}.
$$

This expression points out that the contribution of the shell is proportional to $1 - (r_1/r_2)^3$. Therefore, a thin shell can significantly affect $\mu_p$ only in the case of large value of $|\mu_2|$.

Fig. 8 shows the magnetic “permeability of grain” $\mu_p$ in the case of a magnetic core coated by a nonmagnetic shell. The results indicate that the strong skin effect (large $r_1/\delta$ ratio) substantially reduces $\mu_p$. Only in the case of “fine” powders when $r_1/\delta \ll 1$, the permeability $\mu_p$ restored from experimental data directly corresponds to the intrinsic permeability. In the case of “coarse” powders ($r_1/\delta \gg 1$), the measured $\mu_p$ is much less than the intrinsic one.

5. Dependence on volume fraction of particles

The dependence of the effective parameters $\varepsilon_{eff}$ and $\mu_{eff}$ on the volume fraction $p$ is shown in Fig. 9. In calculations, the powder is assumed to be nonmagnetic. The core conductivity is $\sigma = 5.8 \times 10^7$ S/m, which is the static conductivity of copper. Relative permittivity of the shell is arbitrarily set to $3 + i3$ or $3 + i0.3$. The shell thickness is fixed at 100 nm. The microwave frequency is 2.45 GHz.

Both the real and imaginary parts of $\varepsilon_{eff}$ increases with the volume fraction. The magnitude of the effective permeability $\mu_{eff}$ (especially the imaginary part) changes drastically by orders of magnitude at a percolation threshold. For Bruggeman’s medium the percolation threshold is $p_t = 0.33$. 

![Fig. 7. Permeability $\mu_p$ as a function of intrinsic permeability of the shell $\mu_2$. (a) real part and (b) imaginary part. Core is assumed to be nonmagnetic. To demonstrate the skin effect, ratio $r_1/\delta$ is set to 0.1, 2 and 5. Within the used parameters the resultant $\mu_p$ is a sum of the term depending on $r_1/\delta$ ratio and the term depending on $\mu_2$.](image1)

![Fig. 8. Permeability $\mu_p$ as a function of intrinsic permeability of core $\mu_2$. (a) real part and (b) imaginary part. Shell is assumed to be nonmagnetic. To demonstrate the skin effect, ratio $r_1/\delta$ is set to 0.1, 2 and 5, where $\delta$ is the skin depth of the corresponding nonmagnetic core [see Eq. (10)].](image2)
The effective dielectric permittivity strongly depends on volume fraction (Section 5). The magnitude of \( \varepsilon_{\text{eff}} \) drastically changes at a percolation threshold. Results demonstrate that in the case of highly conductive cores, the loss tangent \( \varepsilon''/\varepsilon' \) of a compacted powder is approximately close to that of the shell when \( p \geq 0.5 \). The effective magnetic permeability changes much smoother with the volume fraction.

The calculations demonstrate that the effective optical parameters of metal powders strongly depend on many parameters: the relative thickness of the shell \( (r_1/r_2) \), the skin effect \( (r_1/\delta) \), parameters of the core and shell, the volume fraction. Ignoring the complex behavior of \( \varepsilon_{\text{eff}} \) and \( \mu_{\text{eff}} \) can lead to the misinterpretation of similar experiments conducted by different groups of researchers. For example, the parameters strongly depend on the thickness of a shell. The shell can be affected by a number of ways: the powder production process, powder storage, the type of gas atmosphere in the heating experiments. This leads to a wide range of magnitudes of the effective permittivity and, as a result, to different coupling of powders with microwaves even if particle size distributions are the same.

Another example concerns with the size of grains. The dependence of the effective parameters on the particle size makes microwave heating sensitive to the size distributions. In the heating experiments, the particle size distributions are frequently characterized by a single number like the average size of grains or the size of passed standard mesh. The obtained results strongly point out that the same average size of grains or that of passed mesh does not guarantee similar coupling of powders with microwaves. In addition, the sensitivity to the particle size suggests the sensitivity to the particle shape. It means that the powders made of spherical grains and those made of flake-shaped grains couple with microwaves differently.

The variety of factors strongly affecting the behavior of microwave heating leads to variety of possible experimental results. To make definite comparison of the results, the detailed description of the experimental conditions is required.

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Appendix A. Expressions for $a_1$ and $b_1$ for a small coated particle

In the case of a coated particle embedded in background medium with the dielectric permittivity $\varepsilon_b$ and magnetic permeability $\mu_b$, the Mie coefficients $a_n$ and $b_n$, where $n = 1, 2, 3, \ldots$, are given by [12]

$$a_n = - R_n^{(1)}(x_2) \frac{m_2 \mu_b H_n^0 - \mu_b D_n^{(1)}(x_2)}{m_2 \mu_b H_n^0 - \mu_b D_n^{(1)}(x_2)},$$

$$b_n = - R_n^{(1)}(x_2) \frac{\mu_b H_n^0 - m_2 H_n^0 D_n^{(1)}(x_2)}{\mu_b H_n^0 - m_2 H_n^0 D_n^{(1)}(x_2)},$$

with

$$R_n^{(2)}(m_2 x_2) D_n^{(2)}(m_2 x_2) - A_n R_n^{(2)}(m_2 x_2) - A_n,$$

$$R_n^{(2)}(m_2 x_2) D_n^{(2)}(m_2 x_2) - B_n R_n^{(2)}(m_2 x_2),$$

and

$$A_n = R_n^{(2)}(m_2 x_1) \frac{m_1 \mu_b D_n^{(1)}(m_1 x_1) - m_2 \mu_1 D_n^{(1)}(m_2 x_1)}{m_1 \mu_b D_n^{(1)}(m_1 x_1) - m_2 \mu_1 D_n^{(1)}(m_2 x_1)},$$

$$B_n = R_n^{(2)}(m_2 x_1) \frac{m_2 \mu_1 D_n^{(1)}(m_1 x_1) - m_1 \mu_1 D_n^{(1)}(m_2 x_1)}{m_1 \mu_b D_n^{(1)}(m_1 x_1) - m_2 \mu_1 D_n^{(1)}(m_2 x_1)}.$$

Here, $x_{1,2} = k_b r_{1,2}$, $k_b = \omega \sqrt{\varepsilon_b \mu_b}$, $\omega = 2\pi f$, $f$ is the frequency of the incident plane wave, $r_{1,2}$ are the radii of the core and the shell; $m_{1,2} = \sqrt{\varepsilon_{1,2} \mu_{1,2} / (\varepsilon_b \mu_b)}$ are the relative refractive indices, with $\varepsilon_{1,2}$ and $\mu_{1,2}$ being the permittivity and permeability of the core and the shell, respectively. The ratios $R_n^{(1)}$ and $R_n^{(2)}$ are defined as

$$R_n^{(1)}(x) = \frac{j_n(x)}{\alpha_n^{(1)}(x)},$$

$$R_n^{(2)}(x) = - \frac{j_n(x)}{\alpha_n^{(2)}(x)},$$

where $j_n$, $y_n$ are the spherical Bessel functions, and $\alpha_n^{(1)}$ is the spherical Hankel functions of the first kind. Logarithmic derivatives are given by

$$D_n^{(1)}(x) = \frac{x \frac{\alpha_{n+1}^{(1)}(x)}{\alpha_n^{(1)}(x)}}{\alpha_n^{(2)}(x)}$$

$$D_n^{(2)}(x) = \frac{x \frac{\alpha_{n+1}^{(2)}(x)}{\alpha_n^{(2)}(x)}}{\alpha_n^{(1)}(x)}$$

where $\alpha_{n+1}^{(1)} = j_{n+1}, \alpha_{n+1}^{(2)} = y_{n+1}$ and $z_{n+1}^{(1)} = h_{n+1}^{(1)}$.

If $|x| < 1$, then only terms with $n = 1$ contribute to the final solution.

Additional condition $|m_2 x_2| < 1$ makes possible the use of the only first terms in Taylor series of $R_n^{(1)}$, $J = 1, 2$ and $D_n^{(1)}, J = 1, 2, 3$, i.e.

$$R_1^{(1)}(x) \approx \frac{x^2}{3}, \quad R_2^{(1)}(x) \approx \frac{x^3}{3}$$

and

$$D_1^{(1)}(x) \approx \frac{2}{x}, \quad D_2^{(1)}(x) \approx D_3^{(1)}(x) \approx - \frac{1}{x}.$$

Here, the argument $x$ can be $x_2$, $m_2 x_2$ or $m_1 x_1$ but not $m_1 x_1$. Since the core is assumed to be highly conductive, its permittivity can be large at microwave frequencies, therefore the full expression for logarithmic derivative $D_n^{(1)}(m_1 x_1)$ should be used:

$$D_1^{(1)}(m_1 x_1) = \frac{2}{m_1 x_1 f_0 (m_1 x_1)},$$

where

$$F_0(y) = \frac{y - \cos y + siny}{y - \cos y - siny + y^2 siny}$$

In this case, $a_1$ and $b_1$ are equal to

$$a_1 \approx \frac{2 \varepsilon_b^2 \mu_0 - \varepsilon_b}{3 \mu_0 + 2 \varepsilon_b},$$

$$b_1 \approx \frac{2 \varepsilon_b^3 \sigma_{\varepsilon_b} - \varepsilon_b}{3 \sigma_{\varepsilon_b} + 2 \varepsilon_b},$$

where $i = -1; \mu_0 = \frac{\mu_2 F_0}{2 + \mu_1 F_0}$ and $\sigma = \sigma_{\varepsilon_b} F_2^2$ with

$$F_2^2 = \frac{1 - (r_1/r_2)^2 F_1^2}{2 + (r_1/r_2)^2 F_1^2},$$

$$F_1^2 = \frac{1 - (\varepsilon_2/\varepsilon_1) F_2^2}{2 + (\varepsilon_2/\varepsilon_1) F_2^2}.$$

The most stringent among the two used conditions ($|x_2| < 1$ and $|m_2 x_2| < 1$) determines the upper limit of appropriate particle size. The same conditions rewritten in terms of wavevectors and particle radius are $|k_b r_{1,2}| < 1$ and $|k_{2,1}| < 1$, where $k_{1,2} = \omega \sqrt{\varepsilon_{1,2} \mu_{1,2}}$.

References